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The $f(\alpha)$ singularity spectrum of the planets in the solar system

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Abstract

The set of planets in the solar system consists of the nine objects Pluto up to and including Jupiter, in increasing order of planet size and mass. We have stimulating numerical evidence that this set can be approximated by a two-scale Cantor multi-fractal with $l_1 \approx 0.40$ and $p_1 \approx 0.15$.

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1. Introduction

We consider the planets of the solar system as the fractal outcome of a particular aggregation process of particles. This process is certainly different from the diffusion limited aggregation (DLA) model [1–5], because the objects are rotating and the internal compactification of the objects is not accounted for, among other things. It is also different from the cluster–cluster model [6] owing to almost the same arguments. We do not propose a theory to describe this specific aggregation process but only furnish some numerical evidence.

Assuming that the distribution of the planet sizes R_c and masses M_c is a two-scale Cantor measure, we develop some theory in section 2 to support the calculation of the scaling values l_1, p_1. In section 3 we show the results, while comments and conclusions are given in section 4.

2. Basic properties of the two-scale multi-fractal

Let $l_1 \leqslant l_2$ be the length rescaling parameters and $p_1 \leqslant p_2$ the measures of a two-scale Cantor multi-fractal (MF hereafter) [7–13] with generator

$$p_1^q l_1^\tau + p_2^q l_2^\tau = 1 \tag{1}$$

so that the partition function on level n is

$$\Gamma_n(q) = (p_1^q l_1^\tau + p_2^q l_2^v)^n = \sum_{i=0}^{2^n-1} \mu_i^q \delta_i^\tau = 1 \tag{2}$$

where $\delta_i = l_1^{n-k} l_2^k$, $\mu_i = p_1^{n-k} p_2^k$ are the size (respectively, mass) of the ith bin in [0, 1] and k is the number of 1's in the binary expansion of i, $i = \sum_{v=1}^{n} a_v 2^{n-v} \in [0, 2^n - 1]$ with $a_v = 0$ or 1. We have used a value of $n = 10$, $2^n = 1024$, throughout this paper.

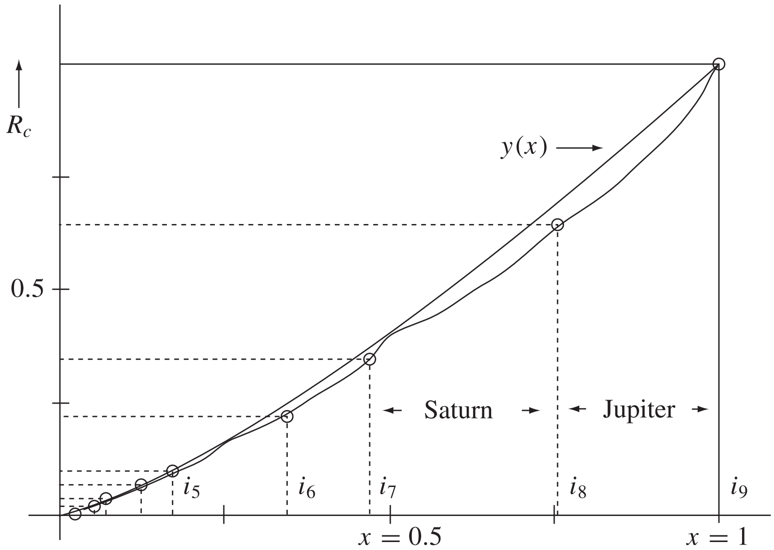

Figure 1. The graph of the measure R_c of equation (3*b*). The broken lines mark the positions of the planets on the curve, i.e. the coarse subdivision $x_k = i_k/2^n$. The full curve is $y(x) = x^g$, where $g = -\ln(l_1)/\ln(2)$.

Furthermore, let $D_0 = 1$ be the dimension of the segment [0, 1], then if $q = 0$ in equation (1) we have

$$l_1^{D_0} + l_2^{D_0} = l_1 + l_2 = 1 \qquad \text{if} \quad q = 1 \quad \text{then} \quad p_1 + p_2 = 1.$$

The expression of the (cumulative) mass distribution from $i = 0$ up to $i = i_j$ is

$$M_{cj} = \sum_{i=0}^{i_j-1} \mu_i = \sum_{i=0}^{i_j-1} p_1^{n-k} p_2^k \tag{3a}$$

and the value at the end is $M_c(i_j = 2^n) = 1$. Similarly, the distribution R_{cj} of the lengths of the radii is determined by

$$R_{cj} = \sum_{i=0}^{i_j-1} \delta_i = \sum_{i=0}^{i_j-1} l_1^{n-k} l_2^k \qquad \text{with} \quad R_c(i_j = 2^n) = 1. \tag{3b}$$

The graph of this measure R_c on level n is shown in figure 1.

After a little algebra we find that, if $i_j = 2^{n-j}$, $j = 0, 1, \ldots, n$, then $R_{cj} = l_1^j$ and $M_{cj} = p_1^j$.

2.1. A coarse subdivision of [0, 1]

Now, suppose the segment [0, 1] is also coarsely subdivided into a set of Np elements according to the set

$$I = \{i_k | k = 1, \ldots, Np \leqslant n, i_0 = 0 < i_1 < \cdots < i_{Np} = 2^n\}.$$

The partition function (2) then becomes

$$\Gamma_n(q) = \sum_{k=1}^{Np} \left(\sum_{i=i(k-1)}^{i(k)-1} \mu_i^q \delta_i^\tau \right) = \sum_{k=1}^{Np} m_k^q r_k^\tau = 1 \tag{4}$$

and produces only the same τ provided that $m_k^q r_k^\tau$ equals the sum $\sum_{i=i(k-1)}^{i(k)-1} \mu_i^q \delta_i^\tau$ for every $k = 1, \ldots, Np$, and $q \in R$.

If $q = 0$ then

$$r_k = R_{ck} - R_{ck-1} = \sum_{i=i(k-1)}^{i(k)-1} \delta_i \tag{5a}$$

is the size of object k and if $q = 1$ then we find

$$m_k = M_{ck} - M_{ck-1} = \sum_{i=i(k-1)}^{i(k)-1} \mu_i \tag{5b}$$

as the mass of object k.

When the values of l_1 and p_1 are known, then we can solve equation (1) in τ and find the exact value τ_0. However, solving equation (2) or (4) in τ, given $\{\delta_k, \mu_k\}$ or $\{r_k, m_k\}$, leads normally to an approximation $\tau = \tau_1$ of τ_0.

If the coarse subdivision is exactly equal to $i_k = 2^{n-Np+k}$ then we obtain $R_{ck} = l_1^{Np-k}$, hence $r_k = l_2 l_1^{Np-k}$, and $M_{ck} = p_1^{Np-k}$, from which $m_k = p_2 p_1^{Np-k}$. The r_k is a power law with respect to the index k in l_1 and so is m_k in p_1. We call this type of subdivision the pure-fractal format. It is obvious that every binomial MF generated by equation (1) has a pure-fractal format; what is not trivial is that the r_k and m_k are power laws with respect to the index k.

2.2. *An exponential linear coarse subdivision*

When the crude subdivision is more stochastic in nature but still exponential linear, that is $i_k = 2^{k_{01}+\beta_k}$, where k_{01} and β are constants, then a fairly good approximation to the distribution is

$$R_{ck} \cong l_1^{n-k_{01}-\beta_k} = R_{c0} a^k \qquad \text{hence} \quad r_k \approx (a-1) R_{ck} \tag{6a}$$

and

$$M_{ck} \cong p_1^{n-k_{01}-\beta_k} = M_{c0} b^k \qquad \text{and} \qquad m_k \approx (b-1) M_{ck}. \tag{6b}$$

Knowing the coarse partition gives us the possibility to acquire the value of l_1 from the value of either R_{c1} or a, and of p_1 from either M_{c1} or b.

2.3. *α and $f(\alpha)$*

The function τ is defined as $\tau = (1-q)D_q$ so that

$$\alpha_q = -\frac{\partial \tau}{\partial q}$$

and $f(\alpha) = \alpha_q q + \tau$. The real alpha is $\alpha_k = \log(m_k)/\log(r_k)$, where the index k is a dummy index of the coarse partition and is not directly related to a power k.

An approximation to this value can be derived by differentiating the equation $\sum_{k=1}^{Np} m_k^q r_k^\tau = 1$ with respect to q, which leads to

$$\alpha_{q,Np} = \sum_{k=1}^{Np} t_k \log(m_k) / \sum_{k=1}^{Np} t_k \log(r_k) \qquad \text{where} \quad t_k = m_k^q r_k^\tau.$$

Note that, if $Np \to \infty$ then the values of τ_1 and $\alpha_{q,Np}$ become exact. This is shown in figure 2, where the $(\alpha, f(\alpha))$ graph is plotted for the values $Np = 5, 9$ and $Np = \infty$.

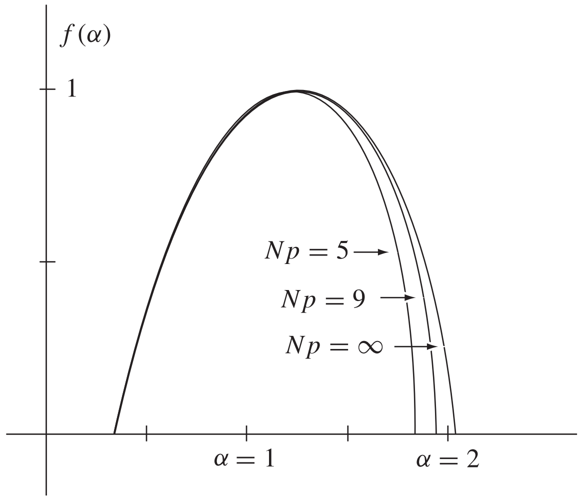

Figure 2. The function $f(\alpha)$ of section 2.3 for different values of Np with $l_1 = 0.40$, $p_1 = 0.155$.

3. Numerical results

Considering the planets as a more or less exact representation of a binomial MF we can make use of the following methods to estimate the values of l_1, p_1, while $D_0 = 1$.

Method 1. The simplest method is to use equations (3*a*) and (3*b*) and to compute R_{cj}, M_{cj} by varying l_1, p_1. The criterion of least squares

$$\sigma^2 = \sum_{j=1}^{Np} (R_{\mathrm{real}\ j} - R_{cj})^2 + (M_{\mathrm{real}\ j} - M_{cj})^2$$

where we denote by $R_{\mathrm{real}\ j}$ the real values in table 1 and by R_{cj} the computed values (3*b*), is then suitable to calculate the set $\{i_j\}$.

The values of l_1, p_1 belonging to the minimum of σ^2 are $l_1 = 0.398$ and $p_1 = 0.154$; the best fit $\{i_j\}$ is shown in table 1.

These values are rather stable; transforming the set once by similarity, $M_c(l_1 R_c(x)) = p_1 M_c(R_c(x))$, or once by affinity, $M_c(l_1 + l_2 R_c(x)) = p_1 + p_2 M_c(R_c(x))$, gives the same values up to a difference of less than 0.01.

Method 2. As appears from equation (4) we can generate a set $S_{Np} = \{(q, \tau(q))\}$ from the set of measurements $\{(r_k, m_k)\}$ by the Newton–Raphson technique [15]. Given the S_{Np} we can compute the values of l_1, p_1 by a variety of possibilities. An easy way is to calculate them from the equations $l_1 + l_2 = 1$ and $p_1 + p_2 = l_1^{\alpha_2} + l_2^{\alpha_1} = 1$.

Taking $\alpha_{\max} = \log(m_{\mathrm{mars}})/\log(r_{\mathrm{mars}}) = 2.043$ and $\alpha_{\min} = 0.328$, related to the planet Jupiter, we find $l_1 = 0.403$ and $p_1 = 0.156$.

Method 3. The values of l_1 and p_1 can be reconstructed from a simple log–linear regression of the values of $\{(R_{cj}, M_{cj})\}$ versus the index k if the coarse partition of section 2.2 is known.

Table 1. Columns 2 and 4 give the normalized cumulative planet radius and mass R_c and M_c (see equations (5)) taken from [14]. In column 3 the percentage deviation (1 − computed R_c/real R_c) × 100 is shown, and the computed R_c is calculated by equation (3*a*) with $l_1 = 0.40$, $p_1 = 0.155$. This is also done in column 5 for the mass. Column 6 shows the associated coarse partition explained in section 2.1. α in column 7 is the real α_k of section 2.3.

Planet	Cumulative size R_c	Dev. R_c (%)	Cumulative mass M_c	Dev. M_c (%)	I_j	α
Pluto	5.591×10^{-3}	17.9	5.733×10^{-5}	−56.6	24	1.88
Mercury	1.773×10^{-2}	6.2	1.810×10^{-4}	1.8	54	2.04
Mars	3.466×10^{-2}	−20.6	4.216×10^{-4}	36.0	71	2.04
Venus	6.480×10^{-2}	−5.5	2.246×10^{-3}	12.1	126	1.80
Earth	9.657×10^{-2}	−3.9	4.484×10^{-3}	8.6	183	1.77
Neptune	2.176×10^{-1}	0.8	4.283×10^{-2}	−31.3	352	1.54
Uranus	3.440×10^{-1}	0.6	7.537×10^{-2}	−0.3	480	1.66
Saturn	6.445×10^{-1}	0.4	2.884×10^{-1}	−1.2	784	1.29
Jupiter	1	—	1	—	1023	0.33

For the coarse partition $\{i_j\}$ of table 1 we have $k_{01} = 4.175$, $\beta = 0.672$, with a linear correlation coefficient $\sigma = 0.996$, so that from $R_{c0} = l_1^{n-k_{01}} = 4.47 \times 10^{-3}$ we compute $l_1 = 0.394$ and from $a = l_1^{-\beta} = 1.863$ we compute $l_1 = 0.395$.

In the same way we obtain from $M_{ck} = M_{c0} b^k$ with $\sigma = 0.996$ and $M_{c0} = 1.41 \times 10^{-5}$ the value $p_1 = 0.146$ and from $b = 0.289$ the value $p_1 = 0.157$.

Unfortunately we are not able to estimate the standard errors in methods 1 and 2, so no comparison of the values is possible.

4. Comments and conclusions

We have demonstrated numerically that the (local) mass and size distribution of the large objects around the Sun can be described approximately by a binomial multi-fractal.

We now make the following observations.

(a) It is difficult to *prove* that the solar system planets are part of a fractal on the basis of a sample of only nine values (r_k, m_k). The theory of section 2 is applicable, but we still need a consistent model. Furthermore, the fractality is also difficult to prove because the values (r_k, m_k) are not exact. Some error sources are:

1. The planets have been subjected to gravitational forces, resonances, etc over millions of years. Therefore, the (r_k, m_k) will not be accurate enough to match a fractal.
2. From section 2.3 $Np < \infty$. It is of no use to improve the accuracy of the l_1, p_1 due to the fact that the finite $Np = 9$ is a large error source (see figure 2).
3. The cut-off values of (r_k, m_k) which define $\alpha_{\min}$ and $\alpha_{\max}$. To give a rather naive illustration, if we add the values of $r_{\text{sun}} = 109.1 R_{\text{earth}}$ and $m_{\text{sun}} = 3.329 \times 10^5 M_{\text{earth}}$ to the set of planetary data then we obtain $\alpha_{\min} = 0.005$ and $\alpha_{\max} = 3.024$. Consequently, the values of l_1, p_1 change, if we maintain the two-scale MF approach. Another star and thus other planets would also yield other values of l_1, p_1.

(b) The values we adopt are: $l_1 \cong 0.40$, $p_1 \cong 0.155$ for a reasonable fit with regard to the real values and α-limits (see figure 3). There is a better fit to the S_{Np} of section 3, method 2, through $l_1 \approx 0.43$, $p_1 \approx 0.17$, but then the approximation to the real values of table 1 is worse. From the set S_{Np} we obtain the values: $\alpha(q = 0) = 1.303$ and $f(q = 0) = D_0 = 1$; $\alpha(q = 1) = 0.724 = f(q = 1) = D_1$; $\alpha(q = 2) = 0.446$

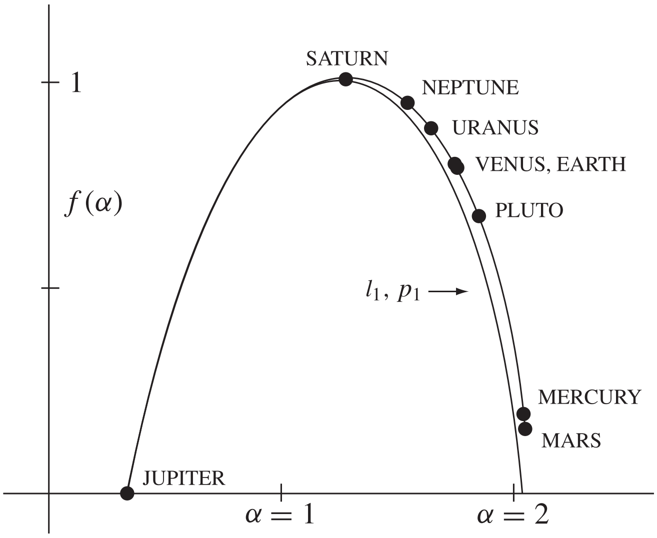

Figure 3. The $f(\alpha)$ of the planets and the fit with $l_1 = 0.40$, $p_1 = 0.155$.

and $f(q = 2) = D_2 = 0.333$; for $q = 0.488$ the value of α is 1 and $f(\alpha = 1)$ is approximately 0.926.

(c) Normally, we would interpret the $\alpha_{min}, \alpha_{max}$ as belonging to the most compact (respectively most rarefied regions of the fractal). Since the planet mass originates from attraction of particles, it is more convincing to relate the $\alpha_{min} = \alpha$(Jupiter) to the maximal and $\alpha_{max} = \alpha$(Mars) to the minimal growth probability. Translated into DLA language, the gravitation around a point is considered to be related to the local growth probability, in this particular fractal aggregation process.

(d) Normalizing the radii of the orbits of the planets and applying logarithmic linear regression, we obtain the value: $R_{orbit,k} = R_0 a^k = 1.766 \times 10^{-3}(1.866)^k$ with a linear correlation coefficient $\sigma = 0.991$. When we compare this to the cumulative R_c of the planets as such (equations (6*a*) and (6*b*)), that is to $R_{ck} = R_{c0}a^k = 4.475 \times 10^{-3}(1.863)^k$, $\sigma = 0.994$, we notice that only the prefactors are different. This suggests that the fractal spatial support (the solar environment) and the results of aggregation (the planets) have a common factor.

At this moment we are still studying this problem (see also [16, 17] for instance).

(e) The procedure of moon formation around a planet is almost similar to that of the planets around the Sun. Also, both processes have as a result a two-dimensional plane with a three-dimensional spherical central mass.

The consequence is a similar $f(\alpha)$, but the numerical approximations of R_c, M_c by equations (3*a*) and (3*b*) or r_k, m_k by (5*a*) and (5*b*) are worse than in the case of the planets.

(f) We conjecture that:

1. The solar system as such is a subset of a multi-fractal, which can be approximated by a two-scale Cantor MF.
2. The interior of the planets is possibly fractal-like with a deviating thermodynamics.

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